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Conditional probability as a decision-making tool: A didactic sequence

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Conditional probability arises as a tool for analyzing a strategy for decision-making that molds to new conditions. From that point of view, an introductory sequence which utilizes diachronic games is designed and analyzed under the framework of didactical engineering, bringing conditional probability into play as a decision-making tool. It can be observed that students tend to base their decisions on heuristics and experiential considerations, and do not see the need for a proper calculation of theoretical probabilities. At most, they use experimentation as a tool, not computing probabilities based on relative frequencies, but comparing absolute frequencies.

Keywords: Decision-making, conditional probability, didactical engineering.

Introduction

UNESCO has dedicated a full chapter about confronting uncertainties on its "Seven complex lessons on education for the future". It describes uncertainties of reality and knowledge, and proposes ways of taking action despite the unavoidable uncertainty of the world (Morin, 1999). Ignoring or downplaying uncertainty could lead us to make fragile decisions, which can generate a negative impact as soon as the circumstances change (Taleb, 2012). In this matter, probability and statistics emerge as the core mathematical subjects for facing this challenge. Thus, these subjects should play an important role in modern mathematics curricula for general education.

However, in general students have low scores on these fields. PISA 2012 reveals that 76.9% of tested students do not pass the second level of accomplishment in the areas of uncertainty and data (Organisation for Economic Co-operation and Development [OECD], 2014). At the best they can apply suitable calculation basic procedures in familiar contexts such as coin tossing or dice rolling. However, they are not able to reason and make critical reflections in order to make valid contextual or general conclusions.

Probability and statistics have given shape to a field of economics studied since the 40s. It has been dominated essentially by "expected value theory" as a normative model of rational choice, proposing that rational individuals maximize the expected value of their utility functions (Friedman & Savage, 1948). This approach has been criticized lately by behavioral psychology and behavioral economics, pointing out many situations in which the axioms of the theory show themselves inadequate for modelling reality (Kahneman & Tversky, 2007). The authors propose a "prospect theory", taking into account individuals' biases towards the probability and impact of each choice.

The research related to this paper is embedded within a broader domain that embraces the relationship between didactics of probabilities and statistics, and decision-making under uncertainty. It involves facing the philosophical debate between the idea of probability and statistics as decision-making tools, against decision-making scenarios as resources for improving the institutionalized techniques within probability and statistics. Moreover, the research relates to critical mathematics education, which emphasizes the empowerment of students as citizens as an argument for mathematics in general education (Skovsmose, 1994).

In this frame, two research questions are addressed. (1) "What kind of probabilistic contents and related didactics could help students make better and more reflexive decisions in their lives?" and (2) "How do we assess the students' learning of probability theory and methods, and which role can decision-making scenarios play in such assessment?"

In particular, this paper reports the results of an exploratory application of didactical engineering that involves conditional probability in decision-making scenarios. The purpose is to illustrate challenges and difficulties involved in the teaching of probability under this point of view.

Theoretical framework

From an *enactivist* perspective (Brown, 2015), knowledge is only reflected by and detectible through action by those who know. One learns with an embodied mind, within a process called *enaction*. This notion breaks *representationist* ideas of the mind, considering an incarnated cognition, in which meanings arise as particular states of cooperation in neuronal networks. These states are put into action –enacted– via retroactive co-definitions between the subjects and the contexts they live in. This concept implies that, when faced upon a learning situation, students will enact the knowledge that had let them to act in similar situations, not only their school experience.

Within the scope of this research, students would enact what they have learned from situations of uncertainty and decisions they made before, their previous formal school knowledge, operational aspects such as heuristics and perceptual aspects like feelings that the context evokes in them.

According to mathematical philosophy literature (e. g. Leitgeb & Hartmann, 2014), two types of decision-making scenarios can be defined. Namely, a *situation under uncertainty* is a setting in which one does not know what the relevant probabilities are, and in decision-making *situation under risk*, the probabilities of the various outcomes are in principle. In both cases, decisions are made based on the best available information and building conjectures about likelihood of different results. We will consider the latter one in this paper. Here we will intend to use mathematical objects, such as probability theory and calculations, in the context of mathematics teaching.

Morin proposes "(…) two ways to confront the uncertainty of action. The first is full awareness of the wager involved in the decision. The second is recourse to strategy" (Morin, 1999, p. 47). Strategy must prevail over program, which gets stuck as soon as the outside conditions are modified. A strategy is meant to be adaptable to variations in the context. In this regard, as one acquires new information about the situation in which one requires to make decisions under uncertainty, the notion of conditional probability lets us incorporate changes in the degrees of belief about possible outcomes (Batanero & Díaz, 2007), improving decision-making based on predictions. As a consequence, the rank of experiments to consider in the classroom becomes wider.

The central mathematical object of study is, therefore, conditional probability. The learning goal is, as stated on the study program at 11th grade in Chile, to "solve problems that involve computation of conditional probabilities within simple situations" (Ministerio de Educación [MINEDUC], 2004). The choice of situations and means of representation (tree diagram, 2x2 tables) are left open for the teachers, but the given established notions involved are:

- Meaning of "probability of event A, given event B", using the notation P(A|B).
- If A and B are independent events, then P(A|B) = P(A).

- If A and B are not independent events, then P(A|B) = P(A and B)/P(B), with P(B) not equal to 0.

As an example, let's say that an experiment consists of tossing a fair coin twice. Let A be the event of obtaining two successive heads, and B be the event of obtaining a head in the first toss. According to previous contents, students should be able to calculate theoretically that P(B) = ½ and P(A and B) = ¼. Given the context, students should be able to interpret and calculate that P(A|B) = ½, because now that we know that B happened, then for A to happen, a second head should be obtained. Also, it may be obtained that P(A|B) = P(A and B)/P(B) = ¼ / ½ = ½.

Methodology

Didactic engineering is assumed as a research and design method, and includes four phases: preliminary analysis, *a priori* analysis, execution, and contrast and redesign (Artigue, 1988).

For the preliminary analysis, historic-epistemological elements are obtained by selected authors, in particular, Pascal and Huygens, and their significant work on the development of probability (Pascal, 1983; Basulto, Camuñez, Ortega, & Pérez, 2004). The analysis is made from a chronological construction of the concepts by the authors and their sociocultural contexts. Practices that build the necessity and give meaning to gambling and decision-making are documented. For the cognitive analysis (Elicer & Carrasco, 2014), exploratory tests are taken to gather productions of students about their probability notions and their strategies to make decisions in the Monty Hall game (Batanero, Fernandes, & Contreras, 2009). The didactic analysis is made from the 11th grade study program and textbook delivered by the Chilean government to public and subsidized schools, which represent 91.1% of the total enrolled students in 2015 (MINEDUC, 2015).

The didactic sequence is then designed taking into account key notions resulting from preliminary analysis and an increasing level of complexity. Students should start making specific calculations and end making justified decisions. For the *a priori* analysis, conjectures emerge from the authors, according to the cognitive analysis. It is fair to anticipate similar outcomes from the students and, therefore, to add questions that help them have a critical insight about them.

The execution stage consists in the application of the didactic sequence to a group of students and the contrasting between initial conjectures and the students' actions and productions. The experimental group is one upper secondary class of 19 students aged 15-17 years old. They have already been introduced to probability calculations using Laplace's law, tree diagrams and basic combinatorial techniques. Students have not yet studied the concept of conditional probability.

Finally, transcriptions of students' written outcomes are tabulated according to defined categories in the preliminary analysis (Elicer & Carrasco, 2014). Those which are unexpected and do not match these categories are highlighted and mentioned in the results. Suggestions for redesign resulting from the discussions with the teachers are mentioned for each activity.

Results and discussion

The designed sequence is fully exposed on the Appendix of this paper. It is meant to be executed as an introduction to the mathematical object of conditional probability. This means no new institutional contents would be presented, they should use their previous knowledge. The first session includes Activities 1 and 2, and the second session concludes with Activities 3 and 4.

For a full revision of relevant elements of the preliminary analysis see Elicer & Carrasco (2016). Those considered for the design are as follows, given in parentheses the questions implemented.

Historic-epistemological. In Pascal-Fermat correspondence, probability analysis arises from projective decision-making, in the effort of setting a fair share (1.1, 1.4, 2.1 and 3.4), in particular when a gambling game stops (2.5, 2.8 and 2.9). After this notion comes the idea of betting with some kind of advantage. Studying the ratio between favorable and all cases comes from getting to know every possible case. Huygens brothers association between forecasting situations, such as life expectancy, and gambling, allows us to give a new meaning to the idea of probability in a game, as an a posteriori calculated probability. From this point of view, possible outcomes of a game are described in statistical data (1.2, 1.4, 2.4, 2.7 and 3.3).

Cognitive. Students conceive that different realizations are all possible cases, without weighing them, and draw upon non-mathematical arguments to make a choice (3.1). They also recognize that they would make different decisions if they played the actual game, where they had to make a choice in situ and not a priori (4.1 and 4.2).

Didactic. The main activity proposed in the textbook is theoretical probability calculation (1.3, 2.2, 2.3, 2.6, 3.2 and 3.5), without decision-making or searching for an advantage in gambling. This is implemented on Activity 3, where the Monty Hall problem involves an actual decision.

Activity 1 goes as expected. Students unanimously recognize this is a fair game because both players have the same probability of winning (1.1), which is well calculated using the Laplace law (1.3). Usually, the distraction of doing eleven repetitions is recognized by them, saying there are too few repetitions, that it is an odd number (1.2), and that results depend on chance or luck (1.2 and 1.4). One particular comment we didn't expect was that "the game is not fair, because the results depend on chance and not on personal abilities" (1.1). Considering a future design, the meaning of fairness should not be trivialized. Might be defined or discussed.

Activity 2 throws similar responses about the basics (2.1 and 2.4), which is the intention. Some students still confuse the concept of "possibility" and "probability" (2.2) when giving their answer, which could be revised before. They do not use their previous combinatorial reasoning. Instead, they count different scenarios than come up to their minds (HTT, HHT, …), which not always lead to counting four possibilities for each player (2.2). For this reason, they might answer that each player has the same probability of winning (2.3), based on their intuition and the scheme made in question 2.2. The same schemes lead some of them to wrong answers.

For the second part of Activity 2, most students recognize that one player has an advantage after the first toss. Just a few could actually compute the theoretical probability (2.6), so most of them base their answers on interpreting experimentation results (2.7). Here the probability of success arises as an estimation of compared absolute or relative frequencies, giving use to fractions. Our observation is that an exact calculation of these probabilities does not seem to be necessary for answering the question, so the intention must be revised.

As for the repartition when the game is interrupted, (2.5, 2.8 and 2.9), there are two main types of reasoning among the students. One big group defends an equal repartition of 50% and 50%, arguing that "even when one of us has more possibilities of winning, randomness says that any of us could win", giving randomness a mean for equality. Others recognize fair to split it according to probabilities, using fractions constructed on their ratio phase (Fandiño, 2015) as an operator to multiply the poll to be shared.

In Activity 3, most of students believe each of the remaining doors give an equal chance of winning (3.1), falling into an isolation effect (Tversky, 1972). Since there is a choice to make, they use personal experience-based explanations, repeating many of the answers obtained on the preliminary analysis. This is expected to change after the experimentation, finding the need for a proper probability calculation. Most of them change their position (3.3 and 3.4), based only on experience. This means the frequentist meaning of probability is stronger than a classic or theoretical one, as a decision-making tool. Questions 3.2 and 3.5 are too confusing for them and most of answers are left blank. We recognize there is no need for analyzing the sample space in an introductory session.

Conclusion

In the context of primary and secondary compulsory education, probability and statistics usually arise as mathematical concepts that represent tools for description of uncertainties. In order to move forward, the authors participate on the idea of having them as elements for decision and action. Didactical sequences should involve escalating decision-making scenarios and questions. According to the historical development of probabilities, it is convenient to ask if a game is fair or not (Hernández, Yumi & de Oliveira, 2010), followed by building a strategy to make it favorable.

Researchers and teachers should anticipate that heuristics and personal experiences are frequently more powerful considerations than calculations about probability and risk, when students are faced with decision-making scenarios. This has been documented not only in the didactics of mathematics research (e.g. Serrano, Batanero, Ortíz & Cañizares, 1998), but also (even previously) in psychology and economics literature. In particular, teaching and learning the conditional probability object could involve decisions within diachronic games. These are subjected to the isolation effect (Tversky, 1972; Kahneman & Tversky, 2007), among other difficulties, such as perceptions of independence and sample space, and interpretations of convergence (Batanero et al., 2009).

We recommend creating new sequences for other probabilistic concepts. Natural extensions are total probabilities and Bayes' theorem. Given information about medical research, students may decide whether approving or not a certain pharmaceutical product; or deciding about changes on their habits according to the relationship between cancer and processed meat or smoking.

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Appendix: Proposed sequence for execution

Activity 1: Single coin toss

Two players choose heads or tails. They toss a coin and whoever guesses wins.

1.1 Do you think this is a fair game? Why?

1.2 Repeat this game eleven times and register who wins each time. Do you keep your answer for question 1?

1.3 What's the probability of winning for each player?

1.4 [Teachers compile the results on the board]. Do you still keep your answer for question 1?

Activity 2: Best out of three

The game consists in two players choosing heads or tails, betting 60 each. They toss a coin successively three times and whoever obtains the most guesses wins.

2.1 Do you think this is a fair game? Why?

2.2 How many options does each player have of winning?

2.3 What's the probability of winning for each player?

2.4 Repeat this game ten times and register who wins each time. Do both players have the same amount of victories? Why do you think this happens?

Now suppose the first coin toss results on heads and the game is interrupted. You must decide what to do with the poll.

2.5 Would it be fair to split the poll by 60 each? Why?

2.6 Could you calculate the probability of winning for each player starting from that point?

2.7 Still assuming the first toss resulted in heads, simulate ten times the two remaining tosses, and register who wins the best out of three each time.

2.8 Given this scenario, would it be fair to split the poll giving 80 to the player who betted heads, and 40 for the one for tails?

2.9 Propose a repartition coherent with each one's probability of winning.

Activity 3: Monty Hall game

You are faced against three doors. Behind two of them there are goats and the other has a new car. Your goal is to guess the door where the car is hidden. The sequence is as follows: (1) The host offers you to pick a door. (2) After your choice, the host opens another door, different from the one you have chosen and shows there's a goat. (3) Now he offers a second chance: will you keep your first choice or change it to the other closed door?

3.1 What would you decide; would you keep your first choice or change it? Explain what is relevant for you to make this decision.

3.2 Which events have the same probability of occurring?

3.3 In pairs, play the game with your cups and car toy. One of the players will always change his or first choice, and the other will never change it. Repeat this ten times and compile the results for the whole class. Is there any difference between both types of players?

3.4 Is, therefore, any way of betting with an advantage?

3.5 Reconsider your answer from question 3.2. Given that the game has two stages of choice, which events have the same probability of occurring?

Activity 4: Plenary

Each pair of students responds the following questions in front of the class.

4.1 How would you face the Monty Hall game if you had to be there?

4.2 What recommendations would you give to someone who is about to play?

References

Artigue, M. (1988). Ingénierie didactique. *Recherches en didactique des mathématiques, 9(3)*, 281−308.

Basulto, J., Camuñez, J., Ortega, F., & Pérez, M. (2004). La correspondencia entre los hermanos Huygens en 1669: Vida media frente a vida mediana. In del Cerro, J.S. (Ed.), *Historia de la probabilidad y la estadística (II)* (pp. 57−69). Madrid: Delta.

Batanero, C., & Díaz, C. (2007). Probabilidad, grado de creencia y proceso de aprendizaje. *Jornadas Nacionales de Enseñanza y Aprendizaje de las Matemáticas, 13*. Granada, Spain.

Batanero, C., Fernandes, J. A., & Contreras, J. M. (2009). Un análisis semiótico del problema de Monty Hall e implicancias didácticas. *Suma: Revista sobre Enseñanza y Aprendizaje de las Matemáticas, 62*, 11−18.

Brown, L. (2015). Researching as an enactivist mathematics education researcher. *ZDM Mathematics Education*, *47*, 185–196.

Elicer, R., & Carrasco, E. (2014). Juegos de azar diacrónicos: un espacio para el encuentro entre las creencias subjetivas y las probabilidades condicionales. In M. Parraguez, H. Rivas, C. Vásquez, N. Pincheira, H. Solar, F. Rojas y E. Chandía (Eds.), *XVIII Jornadas Nacionales de Educación Matemática*, (p. 86). Santiago, Chile: Sociedad Chilena de Educación Matemática.

Elicer, R., & Carrasco, E. (2016). Probabilidad condicional como herramienta para la toma de decisiones. In *Acta Latinoamericana de Matemática Educativa (29)*, (pp. 157–165). México: Comité Latinoamericano de Matemática Educativa A.C.

Fandiño, M. I. (2015). Las fracciones: Aspectos conceptuales y didácticos. In L. A. Hernández Rebollar, J. A. Juárez López, J. Slisko Ignjatov (Eds.), *Tendencias en la educación matemática basada en la investigación*, (pp. 25–38). Puebla, Mexico: BUAP Benemérita Universidad Autónoma de Puebla (Facultad de Ciencias Físico Matemáticas).

Friedman, M., & Savage, L. J. (1948). The utility analysis of choices involving risk. *Journal of the Political Economy*, *56*(4), 279–304.

Hernández, H. M., Yumi, V., & de Oliveira, M. (2010). El uso de juegos para la promoción del razonamiento probabilístico. *UNIÓN: Revista Iberoamericana de Educación Matemática*, *24*, 69–83.

Kahneman, D., & Tversky, A. (2007). Prospect theory: An analysis of decision under risk. *Econometrica*, *47*(2), 263–292.

Leitgeb, H., & Hartmann, S. (2014). *Introduction to mathematical philosophy*. *Chapter 6: Decision*. München, Germany: Ludwig-Maximilians-Universität.

Ministerio de Educación (2004). *Matemática, programa de estudio, tercer año medio, formación general*. Santiago: Unidad de Currículum y Evaluación, Ministerio de Educación, Gobierno de Chile.

Ministerio de Educación (2015). *Variación de matrículas y tasas de permanencia por sector*. (MINEDUC - Evidence Series, No. 30, year 4). Retrieved from http://centroestudios.mineduc.cl/tp_enlaces/portales/tp5996f8b7cm96/uploadImg/File/Evidencias/Evidencias%20final_julio_2015.pdf

Morin, E. (1999). *Seven complex lessons in education for the future*. Paris: UNESCO.

Organisation for Economic Co-operation and Development. (2014). *PISA 2012 results: What students know and can do - Student performance in mathematics, reading and science* (Volume I, Revised edition, February 2014). PISA: OECD Publishing.

Pascal, B. (1983). Cartas de Pascal a Fermat. In C.R. Dampierre (Trad.), *Blaise Pascal: Obras escogidas*. Madrid: Ediciones Alfaguara.

Serrano, L., Batanero, C., Ortíz, J. J., & Cañizares, M. J. (1998). Heurísticas y sesgos en el razonamiento probabilístico de los estudiantes de secundaria. *Educación Matemática, 10*(1), 7–25.

Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education*. Dordrecht, The Netherlands: Kluwer Academic.

Taleb, N. N. (2012). *Antifragile*. New York: Random House.

Tversky, A. (1972). Elimination by aspects: A theory of choice. *Psychological Review*, *79*(4), 281–299.